

The moduli space of hypersurfaces whose singular locus has high dimension

Kaloyan Slavov

August 7, 2012

Abstract

Let k be an algebraically closed field and let b and n be integers with $n \geq 3$ and $1 \leq b \leq n - 1$. Consider the moduli space X of hypersurfaces in \mathbb{P}_k^n of fixed degree l whose singular locus is at least b -dimensional. We prove that for large l , X has a unique irreducible component of maximal dimension, consisting of the hypersurfaces singular along a linear b -dimensional subspace of \mathbb{P}^n . The proof will involve a probabilistic counting argument over finite fields.

Contents

1	Introduction	1
2	Notation	3
3	On the dimension of the space of hypersurfaces which contain a fixed integral closed subscheme	4
3.1	An upper bound on the dimension of the space of F such that $C \subset V(F)$, for a fixed C	4
3.2	Final preparations	6
4	The case of large degree d, when $k = \overline{\mathbb{F}_p}$	7
4.1	Reduction to a problem over finite fields	8
4.2	The key step (large degree d)	8
5	Proof of the main theorem	10
5.1	Restatement of the problem and the end of the proof	10
5.2	Uniqueness of the largest component (in characteristic 0)	12
5.3	The second largest component	13

1 Introduction

Let n and b be fixed integers with $n \geq 3$ and $1 \leq b \leq n - 1$, and let k be an algebraically closed field of characteristic $p \geq 0$. Fix a positive integer l . Inside the projective space

of all hypersurfaces in \mathbb{P}^n of degree l , consider the ones which are singular along some b -dimensional closed subscheme,

$$X = \{[F] \in \mathbb{P}(k[x_0, \dots, x_n]_l) \mid \dim V(F)_{\text{sing}} \geq b\}$$

(this is a closed subset).

A simple argument (Lemma 5.1 in [5]) shows that

$$X^1 := \{[F] \in X \mid L \subset V(F)_{\text{sing}} \text{ for some linear } b\text{-dimensional } L \subset \mathbb{P}^n\}$$

is an irreducible closed subset of X of dimension $\binom{l+n}{n} - a_{n,b}(l)$, where

$$\begin{aligned} a_{n,b}(l) &:= \binom{l+b}{b} + (n-b) \binom{l-1+b}{b} + 1 - (b+1)(n-b) \\ &= \frac{n-b+1}{b!} l^b + \dots \end{aligned}$$

The goal of this paper is the following

Theorem 1.1. *There exists an integer $l_0 = l_0(n, b, p)$, such that for all $l \geq l_0$, X^1 is the unique irreducible component of X of maximal dimension.*

In fact, the proof of the theorem will give a simple procedure to compute a possible value of l_0 , given n, b, p (assuming a conjecture of Eisenbud and Harris when $b \geq 2$). In addition, again for large l , we find the second largest component of X , at least when $\text{char } k \neq 0$: it comes from the hypersurfaces singular along an integral closed subscheme of degree 2 (Corollary 5.11).

We now sketch the main idea of the proof. In [5], we proved that there is a projective variety $T^d \rightarrow \text{Spec } \mathbb{Z}$ such that the set of closed points of the basechange $T_k^d = T^d \times \text{Spec } k$ is the set of all hypersurfaces $[F] \in \mathbb{P}(k[x_0, \dots, x_n]_l)$ whose singular locus contains a subscheme with Hilbert polynomial among the Hilbert polynomials $\{P_\alpha\}$ of integral b -dimensional closed subschemes of \mathbb{P}^n of degree d . The goal is to bound $\dim T^d$ by $\dim X^1$, since X is a finite union

$$X = \bigcup_{d=1}^{l(l-1)^{n+1}} T_k^d$$

(as we will easily see later in Lemma 5.1). The first step towards Theorem 1.1 is the case of “small” degree $d \leq \frac{l+1}{2}$; this was accomplished in [5], where we proved the following

Theorem 1.2. *There exists $l_0 = l_0(n, b)$ such that for $l \geq l_0$ and $2 \leq d \leq \frac{l+1}{2}$, any irreducible component Z of T_k^d satisfies $Z = X^1$ or $\dim Z < \dim X^1$.*

(The proof of this relied on a conjecture of Eisenbud and Harris when $b \geq 2$; see Section 5.2 in [5].)

The second step is to handle the case $d \geq \frac{l+1}{2}$ (“large” degree), which we accomplish in the present article. For this, the first main observation is that it suffices to assume that $k = \overline{\mathbb{F}}_p$ in the statement of the main theorem. The reason is that the variety T_k^d comes via basechange from a universal projective variety $T^d \rightarrow \text{Spec } \mathbb{Z}$, and in order to give an

upper bound for $\dim T_{\mathbb{Q}}^d$, by upper-semicontinuity, it suffices to give an upper bound for $\dim T_{\mathbb{F}_p}^d$ for a single prime p (we will take $p = 2$).

So let $k = \overline{\mathbb{F}_p}$ and $d \geq \frac{l+1}{2}$. We have to give an upper bound for the dimension of

$$T^d = \{[F] \in \mathbb{P}(k[x_0, \dots, x_n]_l) \mid V(F)_{\text{sing}} \text{ contains} \\ \text{a subscheme with Hilbert polynomial among } \{P_\alpha\}\}.$$

Any variety T over $\overline{\mathbb{F}_p}$ comes from a variety T_0 defined over some finite field \mathbb{F}_{q_0} ; in order to give an upper bound $\dim T \leq A$, it suffices to prove that $\#T_0(\mathbb{F}_q) = O(q^A)$ as $q \rightarrow \infty$, by the result of Lang-Weil [3]. So we reduce the problem to giving an upper bound on the number of hypersurfaces $F \in \mathbb{F}_q[x_0, \dots, x_n]_l$ such that $V(F)_{\text{sing}}$ contains an integral closed subscheme of large degree d .

For this, we mimic the main argument in [4]. We sketch it here in the case $b = 1$ and $l \equiv 1 \pmod{p}$ to simplify notation. Write F in the form

$$F = F_0 + \sum_{i=0}^n G_i^p x_i,$$

where F_0 has degree l , each G_i has degree $\tau = \frac{l-1}{p}$, and note that

$$\frac{\partial F}{\partial x_i} = \frac{\partial F_0}{\partial x_i} + G_i^p.$$

Fix F_0 . We exhibit a large supply of (G_0, \dots, G_n) such that the F constructed in this way has the property that $V(F)_{\text{sing}}$ contains no integral curves of degree d . To do this, we first give a large supply of (G_0, \dots, G_{n-2}) such that $V(\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_{n-2}})$ has all components of dimension 1. The number of such components is bounded by Bézout's theorem. It remains to give a large supply of G_{n-1} such that no irreducible component C of $V(\frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_{n-2}})$ of degree d is contained in $V(\frac{\partial F_0}{\partial x_0} + G_{n-1}^p)$. We accomplish this by specializing C to a union of d lines (as in our preceding paper [5]), and giving an upper bound on the number of G_{n-1} with $C \subset V(\frac{\partial F_0}{\partial x_0} + G_{n-1}^p)$. With some technical details concerning the uniqueness of the largest-dimensional component in characteristic 0, this completes the proof of Theorem 1.1. The discussion of the second largest component is along the same lines.

2 Notation

For a field k , the graded ring $k[x_0, \dots, x_n]$ will be denoted by S . For a graded S -module M (in particular, for a homogeneous ideal), M_l will denote the l -th graded piece of M . When $I \subset S$ is a homogeneous ideal, $(I^2)_l$ is denoted simply by I_l^2 . Also, $k[x_0, \dots, x_n]_{\leq l}$ denotes the vector space of (inhomogeneous) polynomials whose total degree is at most l . When the field k and the integer l are fixed, V will denote the vector space $V = k[x_0, \dots, x_n]_l$.

For a finite-dimensional k -vector space V , $\mathbb{P}(V)$ denotes the projective space parametrizing lines in V . Given a homogeneous ideal $I \subset k[x_0, \dots, x_n]$, $V(I)$ denotes the closed subscheme $\text{Proj}(k[x_0, \dots, x_n]/I) \hookrightarrow \mathbb{P}_k^n$, and for $i = 0, \dots, n$, $D_+(x_i)$ is the complement of

$V(x_i)$. We often abbreviate $V(\{G_i\}_{i \in I}) \subset \mathbb{P}^n$ as $V(G_i)$, when the index set I is irrelevant or understood.

For $F \in S_l$, $V(F)_{\text{sing}} \subset \mathbb{P}^n$ is the closed subscheme $V(F, \frac{\partial F}{\partial x_i}) = V(F, \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n})$ of \mathbb{P}^n , so when $F \neq 0$, the underlying topological space of $V(F)_{\text{sing}}$ is the singular locus of $V(F)$.

If $C \hookrightarrow \mathbb{P}^n$ is a closed subscheme of dimension b and Hilbert polynomial $P_C(z) = \frac{d}{b!}z^b + \dots$, we say that C has degree d .

We will reuse l_0 for different bounds as we go along, in order to avoid unnecessary notation; however, it will be clear that we are actually referring to different values of l_0 even though we use the same symbol. Also, it will be understood that sometimes the value of l_0 is the maximum of a finite set of previously defined bounds, each of them still denoted by l_0 .

When X is a scheme of finite type over an algebraically closed field, we often identify X with its set of closed points, since most of our arguments will be just on the level of closed points. So when we say “ $x \in X$,” we usually refer to a closed point $x \in X$ (this will be clear from the context).

3 On the dimension of the space of hypersurfaces which contain a fixed integral closed subscheme

This section is the preparation for the main arguments in the paper. The main result of 3.1 is stated in Corollaries 3.4 and 3.6 in a form that is most convenient for Corollary 3.8, which is the goal of this section.

3.1 An upper bound on the dimension of the space of F such that $C \subset V(F)$, for a fixed C

Let $V = k[x_0, \dots, x_n]_l$, where k is a fixed algebraically closed field.

Lemma 3.1. *Fix positive integers l, m , with $m \leq l+1$. For any integral closed subscheme $C \subset \mathbb{P}^n$ of dimension b and degree $d \geq m$, we have*

$$\text{codim}_V\{G \in V \mid C \subset V(G)\} \geq \sum_{e=1}^m \binom{l-e+1+b}{b} =: A_b(l, m).$$

Proof. We apply the specialization technique from Section 4 in [5] and follow closely the argument in Lemma 4.5 there. Namely, we specialize C to a union of d b -dimensional linear spaces L_1, \dots, L_d containing a common $(b-1)$ -dimensional linear subspace P . Without loss of generality, assume that $P = V(x_0, x_{b+1}, \dots, x_n)$ and that none of the L_i is contained in the hyperplane $V(x_0)$, so for uniquely determined tuples $(p_{b+1}^{(i)}, \dots, p_n^{(i)}) \in k^{n-b}$, the ideal I_i of L_i has the form

$$I_i = (x_{b+1} - p_{b+1}^{(i)}x_0, \dots, x_n - p_n^{(i)}x_0), \quad i = 1, \dots, d-1$$

and without loss of generality

$$I_d = (x_{b+1}, \dots, x_n).$$

Then as in Section 4 in [5], we have

$$\text{codim}_V\{G \in V \mid C \subset V(G)\} \geq \text{codim}_V\{G \in V \mid L_1 \cup \cdots \cup L_d \subset V(G)\}.$$

Throwing away some of these linear spaces if necessary, we may assume $d = m$. So we induct on $m = 1, \dots, l+1$ to give a lower bound for $\mu_m := \dim(S/I_1 \cap \cdots \cap I_m)_l$. We have a short exact sequence

$$0 \rightarrow \left(\frac{I_1 \cap \cdots \cap I_{m-1}}{I_1 \cap \cdots \cap I_m} \right)_l \rightarrow (S/I_1 \cap \cdots \cap I_m)_l \rightarrow (S/I_1 \cap \cdots \cap I_{m-1})_l \rightarrow 0.$$

Consider $F = \prod_{i=1}^{m-1} (x_{m_i} - p_{m_i}^{(i)} x_0)$ and note that the elements $FP \in (I_1 \cap \cdots \cap I_{m-1}/I_1 \cap \cdots \cap I_m)_l$, where P runs through a basis of $k[x_0, \dots, x_b]_{l-m+1}$ are linearly independent. This gives a lower bound for the dimension of the kernel in the above short exact sequence, and hence

$$\mu_m \geq \mu_{m-1} + \binom{l-m+1+b}{b},$$

which proves the statement of the Lemma by induction. \square

Remark 3.2. Note that

$$A_b(l, l+1) \geq \sum_{e=1}^{\lceil \frac{l+1}{2} \rceil} \binom{l-e+1+b}{b} \geq \frac{l+1}{2} \binom{\frac{l}{2}+b}{b},$$

so $A_b(l, l+1)$ dominates a polynomial in l of degree $b+1$.

Corollary 3.3. *Let k be an algebraically closed field, and $k_0 \subset k$ a subfield. Again, let m, l be fixed integers, with $m \leq l+1$. Let $C \subset \mathbb{P}_k^n$ be a b -dimensional integral closed subscheme (not necessarily defined over k_0) of degree $d \geq m$. Then*

$$\text{codim}_{k_0[x_0, \dots, x_n]_l} \{G \in k_0[x_0, \dots, x_n]_l \mid C \subset V(G)\} \geq A_b(l, m).$$

Here, the condition $C \subset V(G)$ (inclusion of closed subschemes of \mathbb{P}_k^n) makes sense when we regard $G \in k[x_0, \dots, x_n]_l$ first.

Proof. It suffices to prove that

$$\dim_{k_0} \{G \in k_0[x_0, \dots, x_n]_l \mid C \subset V(G)\} \leq \dim_k \{G \in k[x_0, \dots, x_n]_l \mid C \subset V(G)\}.$$

This is automatic, since any k_0 -linearly independent elements in $k_0[x_0, \dots, x_n]_l$ are k -linearly independent in $k[x_0, \dots, x_n]_l$. \square

Corollary 3.4. *Let $k_0 = \mathbb{F}_q$ now. Let $C \subset \mathbb{P}_{\mathbb{F}_p}^n$ be an integral b -dimensional closed subscheme of degree $d \geq m$ (again, $m \leq l+1$ is fixed). For G chosen randomly from $\mathbb{F}_q[x_0, \dots, x_n]_l$, we have*

$$\text{Prob}(C \subset V(G)) \leq q^{-A_b(l, m)}.$$

Proof. This is just a restatement of Corollary 3.3, since

$$\#\{G \in \mathbb{F}_q[x_0, \dots, x_n]_l \mid C \subset V(G)\} = q^{\dim\{G \mid C \subset V(G)\}}. \quad \square$$

Lemma 3.5. *Let k be an algebraically closed field, and $S \subset \mathbb{P}_k^n$ an integral closed subscheme of dimension at least $b + 1$. Then*

$$\text{codim}_{k[x_0, \dots, x_n]_l} \{G \in k[x_0, \dots, x_n]_l \mid S \subset V(G)\} \geq \binom{l+b+1}{b+1}.$$

Proof. We can assume that $\dim S = b + 1$. This is a particular case of Lemma 3.1; just note that $A_{b+1}(l, 1) = \binom{l+b+1}{b+1}$. \square

The same argument leading from Lemma 3.1 to Corollary 3.4 leads from Lemma 3.5 to the following

Corollary 3.6. *Let $k_0 = \mathbb{F}_q$ now. Let $S \subset \mathbb{P}_{\mathbb{F}_p}^n$ be an integral closed subscheme of dimension at least $b + 1$. For G chosen randomly from $\mathbb{F}_q[x_0, \dots, x_n]_l$, we have*

$$\text{Prob}(S \subset V(G)) \leq q^{-\binom{l+b+1}{b+1}}.$$

3.2 Final preparations

Consider the natural homogenization map $\sim: \mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq l} \xrightarrow{\sim} \mathbb{F}_q[x_0, \dots, x_n]_l$ with respect to the variable x_n . We have to be slightly careful because this is not the usual homogenization map (which takes a polynomial and homogenizes it to the smallest possible degree); we think of \sim as “homogenization-to-degree- l ” map. Set $\tau = \lfloor \frac{l-1}{p} \rfloor$.

Lemma 3.7. *Let $Z \subset \mathbb{P}_{\mathbb{F}_p}^n$ be an integral closed subscheme not contained in the hyperplane $V(x_n)$. Let $F_0 \in \mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq l-1}$ be a fixed polynomial. Then, as G is chosen randomly from $\mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq \tau}$, we have*

$$\text{Prob}(Z \subset V((F_0 + G^p)^\sim)) \leq \text{Prob}(Z \subset V(G^\sim)).$$

Here, the first \sim is homogenization to degree $l - 1$, and the second one is homogenization to degree τ .

Proof. Let $I \subset \overline{\mathbb{F}_p}[x_0, \dots, x_{n-1}]$ be the (radical) ideal of $Z \cap D_+(x_n) \subset D_+(x_n)$. We claim that for an inhomogeneous polynomial $H \in \mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq l-1}$, we have $Z \subset V(H^\sim)$ if and only if $H \in I$. For this, first notice that $V(H^\sim)$ is either $V(H)^-$ or $V(H)^- \cup V(x_n)$ (where $V(H)^-$ is the topological closure of $V(H) \subset D_+(x_n)$ in $\mathbb{P}_{\mathbb{F}_p}^n$), depending on whether or not the degree of H is equal to the degree of homogenization of the map \sim . Since Z is irreducible and not contained in $V(x_n)$, we have $Z \subset V(H^\sim)$ if and only if $Z \subset V(H)^-$. In turn, since $Z \cap D_+(x_n) \neq \emptyset$, this condition is equivalent to $Z \cap D_+(x_n) \subset V(H)$, which is precisely the condition $H \in I$.

Therefore, $Z \subset V((F_0 + G^p)^\sim)$ if and only if $F_0 + G^p \in I$. If $F_0 + G^p \in I$ and $F_0 + G_1^p \in I$, then $(G - G_1)^p \in I$, and hence $G' := G - G_1 \in I$. So the number of G with $F_0 + G^p \in I$ is either zero, or is equal to the number of elements $G' \in I$ with $G' \in \mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq \tau}$. This is precisely the number of $G' \in \mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq \tau}$ such that $Z \subset V((G')^\sim)$. \square

Corollary 3.8. *Keep the notation of Lemma 3.7.*

a) *If $\dim Z \geq b + 1$, then*

$$\text{Prob}(Z \subset V((F_0 + G^p)^\sim)) \leq q^{-\binom{\tau+b+1}{b+1}}.$$

b) *If $\dim Z = b$ and $\deg Z = d \geq m$, then*

$$\text{Prob}(Z \subset V((F_0 + G^p)^\sim)) \leq q^{-A_b(\tau, m')},$$

where $m' = \min(m, \tau + 1)$.

Proof. Combine Lemma 3.7 with Corollaries 3.4 and 3.6. □

4 The case of large degree d , when $k = \overline{\mathbb{F}_p}$

This section is the heart of the present paper. Fix n and $1 \leq b \leq n - 1$ as usual, and fix a prime p . Let $k = \overline{\mathbb{F}_p}$. Recall the definition of $A_b(l, m)$ from Section 3.1 and the definition of T_k^d from the Introduction.

The goal of this section is to handle the case of large d when $k = \overline{\mathbb{F}_p}$. Specifically, we prove the following

Proposition 4.1. *Fix a triple of positive integers (l, m, a) . Set $\tau = \lfloor \frac{l-1}{p} \rfloor$ and $m' = \min(m, \tau + 1)$. Suppose that*

$$\binom{\tau + b + 1}{b + 1} > a - 1 \quad \text{and} \quad A_b(\tau, m') > a - 1.$$

Let $d \geq m$. If Z is an irreducible component of T_k^d , then either $Z \subset T_k^{d'}$ for some $1 \leq d' < d$, or

$$\dim Z \leq \binom{l + n}{n} - a.$$

Let $Z \subset T_k^d$ be an irreducible component (notation and assumptions as above). Suppose that $Z \not\subset \bigcup_{d'=1}^{d-1} T_k^{d'}$. Then $Z - \left(Z \cap \left(\bigcup_{d'=1}^{d-1} T_k^{d'} \right) \right) \subset Z$ is a dense open subset, and therefore is of the same dimension as Z . It is contained in

$$\hat{T}^d := T_k^d - \left(T_k^d \cap \left(\bigcup_{d'=1}^{d-1} T_k^{d'} \right) \right).$$

So the goal is now to prove that $\dim \hat{T}^d \leq \binom{l+n}{n} - a$.

4.1 Reduction to a problem over finite fields

We begin with a general discussion, which applies to any (quasiprojective) variety over $\overline{\mathbb{F}_p}$. Let $T = \cap V(G_i) - \cap V(G'_j) \subset \mathbb{P}_{\overline{\mathbb{F}_p}}^M$ be a quasiprojective variety over $\overline{\mathbb{F}_p}$, where $G_i, G'_j \in \overline{\mathbb{F}_p}[y_0, \dots, y_M]$. Let A be an integer, and suppose we want to prove that $\dim T \leq A$. There is a finite field \mathbb{F}_{q_0} such that $G_i, G'_j \in \mathbb{F}_{q_0}[y_0, \dots, y_M]$, so T comes from $T_0 := \cap V(G_i) - \cap V(G'_j) \subset \mathbb{P}_{\mathbb{F}_{q_0}}^M$, which is now a variety over \mathbb{F}_{q_0} . We know that $\dim T = \dim T_0$, so suffices to prove that $\dim T_0 \leq A$. For this, by the result of Lang-Weil [3], it suffices to prove that $\#T_0(\mathbb{F}_q) = O(q^A)$ as $q \rightarrow \infty$ (through powers of q_0 of course).

Consider now $T = \hat{T}^d \subset \mathbb{P}(\overline{\mathbb{F}_p}[x_0, \dots, x_n]_l)$, and let \hat{T}_0^d (a variety over a finite field \mathbb{F}_{q_0}) be as in the previous paragraph. In particular, $\hat{T}_0^d(\mathbb{F}_q)$ consists of all $[F] \in (\mathbb{F}_q[x_0, \dots, x_n]_l - \{0\})/\mathbb{F}_q^*$ such that when we regard $[F]$ in $(\overline{\mathbb{F}_p}[x_0, \dots, x_n]_l - \{0\})/\overline{\mathbb{F}_p}^*$, we have that $[F] \in \hat{T}^d \subset \mathbb{P}(\overline{\mathbb{F}_p}[x_0, \dots, x_n]_l)$.

Remark 4.2. Even if F has coefficients in \mathbb{F}_q , we always consider $V(F)$ and $V(F)_{\text{sing}}$ as subschemes of $\mathbb{P}_{\overline{\mathbb{F}_p}}^n$ by first regarding F in $\overline{\mathbb{F}_p}[x_0, \dots, x_n]$.

It is easy to see (see the argument in the proof of Corollary 5.6 in [5]) that the set $\hat{T}_0^d(\mathbb{F}_q)$ is a subset of

$$\begin{aligned} \tilde{T}^d := \{[F] \in (\mathbb{F}_q[x_0, \dots, x_n]_l - \{0\})/\mathbb{F}_q^* \mid V(F)_{\text{sing}} \subset \mathbb{P}_{\overline{\mathbb{F}_p}}^n \text{ contains} \\ \text{an integral } b\text{-dimensional subscheme (over } \overline{\mathbb{F}_p}) \text{ of degree } d\}. \end{aligned}$$

So our goal now is to prove that $\#\tilde{T}^d = O(q^{\binom{l+n}{n}-a})$ as $q \rightarrow \infty$ (through powers of q_0).

As F is chosen randomly from $\mathbb{F}_q[x_0, \dots, x_n]_l$, let Λ be the event that $V(F)_{\text{sing}}$ contains an *integral* b -dimensional subscheme of degree d . Thus, our task is to prove that $\text{Prob}(\Lambda)q^{\binom{l+n}{n}} = O(q^{\binom{l+n}{n}-a+1})$, or equivalently, that $\text{Prob}(\Lambda) = O(q^{-a+1})$ as $q \rightarrow \infty$ (through powers of q_0).

4.2 The key step (large degree d)

Fix a triple (l, m, a) of positive integers. Recall that $\tau = \lfloor \frac{l-1}{p} \rfloor$ and $m' = \min(m, \tau + 1)$. Let $d \geq m$.

As F^\sim is chosen randomly from $\mathbb{F}_q[x_0, \dots, x_n]_l$, or, equivalently, as F is chosen randomly from $\mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq l}$, let E_n be the event that the following two conditions are satisfied:

- For each $i = 0, \dots, n - b - 1$, the variety $V(\frac{\partial F^\sim}{\partial x_0}, \dots, \frac{\partial F^\sim}{\partial x_i})$ has all irreducible components of dimension $n - i - 1$, except possibly for components contained in the hyperplane $V(x_n)$.
- If $C \subset V(\frac{\partial F^\sim}{\partial x_0}, \dots, \frac{\partial F^\sim}{\partial x_{n-b-1}})$ is a b -dimensional integral closed subscheme of degree d , then either $C \subset V(x_n)$, or $C \not\subset V(\frac{\partial F^\sim}{\partial x_{n-1}})$.

We now proceed to bound $\text{Prob}(E_n)$ from below (this is the hard part).

Lemma 4.3.

$$\text{Prob}(E_n) \geq \left(\prod_{i=0}^{n-b-1} \left(1 - \frac{(l-1)^i}{q^{\binom{\tau+b+1}{b+1}}} \right) \right) \left(1 - \frac{(l-1)^{n-b}}{q^{A_b(\tau, m')}} \right). \quad (1)$$

Proof. We now mimic the main argument in [4, Section 2.3]. We will generate a random F by choosing $F_0 \in \mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq l}$, $G_i \in \mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq \tau}$ randomly, in turn, and then setting

$$F := F_0 + G_0^p x_0 + \dots + G_{n-1}^p x_{n-1}. \quad (2)$$

For $F \in \mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq l-1}$, the number of tuples $(F_0, G_0, \dots, G_{n-1})$ for which (2) holds is independent of F . We have

$$\frac{\partial F}{\partial x_i} = \frac{\partial F_0}{\partial x_i} + G_i^p.$$

Moreover, the homogenization map \sim commutes with differentiation, so

$$\frac{\partial F^\sim}{\partial x_i} = \left(\frac{\partial F_0}{\partial x_i} + G_i^p \right)^\sim$$

(again, the two uses of \sim here refer to homogenizations to different degrees, l and $l-1$, respectively).

Let $i \in \{0, \dots, n-b-1\}$. Suppose that F_0, G_0, \dots, G_{i-1} are fixed such that $V(\frac{\partial F^\sim}{\partial x_0}, \dots, \frac{\partial F^\sim}{\partial x_{i-1}})$ has only $(n-i)$ -dimensional components, except possibly for components contained in the hyperplane $V(x_n)$. By Bézout's theorem (see p. 10 in [1] for the version we are using here), $V(\frac{\partial F^\sim}{\partial x_0}, \dots, \frac{\partial F^\sim}{\partial x_{i-1}})$ has at most $(l-1)^i$ irreducible components. Let Z be one of them, and suppose that $Z \not\subset V(x_n)$. As G_i is chosen randomly from $\mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq \tau}$, we claim that

$$\text{Prob} \left(Z \subset V \left(\frac{\partial F^\sim}{\partial x_i} \right) \right) \leq q^{-\binom{\tau+b+1}{b+1}}.$$

This follows from Corollary 3.8a, since $\dim Z = n-i \geq b+1$.

For the final step, conditioned on a choice of $F_0, G_0, \dots, G_{n-b-1}$ such that $V(\frac{\partial F^\sim}{\partial x_0}, \dots, \frac{\partial F^\sim}{\partial x_{n-b-1}})$ has only b -dimensional components, except possibly for components contained in $V(x_n)$, we claim that the probability, as $G_{n-1} \in \mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq \tau}$, that some b -dimensional component C of $V(\frac{\partial F^\sim}{\partial x_0}, \dots, \frac{\partial F^\sim}{\partial x_{n-b-1}})$ of degree d and not contained in $V(x_n)$, is contained in $V(\frac{\partial F^\sim}{\partial x_{n-1}})$, is at most $(l-1)^{n-b} q^{-A_b(\tau, m')}$.

Indeed, the number of b -dimensional components C of $V(\frac{\partial F^\sim}{\partial x_0}, \dots, \frac{\partial F^\sim}{\partial x_{n-b-1}})$ of degree d is at most $(l-1)^{n-b}$, by Bézout's theorem again (this is a bound on the total number of components of all dimensions). If we fix a b -dimensional component C of degree d and not contained in $V(x_n)$, for fixed $F_0, G_0, \dots, G_{n-b-1}$, the probability (as G_{n-1} is chosen randomly from $\mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq \tau}$) that $C \subset V \left(\left(\frac{\partial F_0}{\partial x_{n-1}} + G_{n-1}^p \right)^\sim \right)$, is at most $q^{-A_b(\tau, m')}$, by Corollary 3.8b. \square

Proof of Proposition 4.1. By the hypothesis of Proposition 4.1, each of the exponents on the right hand side of (1) is greater than $a-1$. By virtue of the inequality $\prod (1 - \varepsilon_i) \geq 1 - \sum \varepsilon_i$, Lemma 4.3 implies that $\text{Prob}(E_n) \geq 1 - \frac{1}{q^{a-1}}$ for large q . Therefore,

$$1 - \text{Prob}(E_n) = O \left(\frac{1}{q^{a-1}} \right) \quad \text{as } q \rightarrow \infty.$$

As $F \in \mathbb{F}_q[x_0, \dots, x_{n-1}]_{\leq l}$, let E'_n be the event that any integral b -dimensional closed subscheme $C \subset V(F)_{\text{sing}}$ of degree d is contained in $V(x_n)$. Then E_n implies E'_n . For each $i = 0, \dots, n-1$, define E_i, E'_i in analogy with E_n, E'_n , except with dehomogenization with respect to the variable x_i (and any ordering of the remaining variables). The same conclusion $1 - \text{Prob}(E_i) = O(\frac{1}{q^{a-1}})$ holds for all $i = 0, \dots, n$. Note that Λ (defined at the end of Section 4.1) implies $\bigcup_{i=0}^n \overline{E'_i}$, where $\overline{E'_i}$ denotes the event opposite to E'_i . Indeed, $\bigcap V(x_i) = \emptyset$, so we cannot have $C \subset V(F)_{\text{sing}}$ contained in all the coordinate hyperplanes. Therefore,

$$\text{Prob}(\Lambda) \leq \sum_{i=0}^n (1 - \text{Prob}(E'_i)) \leq \sum_{i=0}^n (1 - \text{Prob}(E_i)) = O\left(\frac{1}{q^{a-1}}\right) \text{ as } q \rightarrow \infty,$$

as desired. \square

5 Proof of the main theorem

We now put together the main results Theorem 1.2 (proved in [5]; recall that this assumes a conjecture of Eisenbud and Harris when $b \geq 2$) and Proposition 4.1 from the previous section and finish the proof of Theorem 1.1. Namely, Theorem 1.1 follows immediately from our previous work when $k = \overline{\mathbb{F}_p}$, and we use upper-semicontinuity applied to $T^d \rightarrow \text{Spec } \mathbb{Z}$ to prove the case $\text{char } k = 0$ (Section 5.1). However, there are technicalities (Corollary 5.7) concerning the uniqueness of the largest component in characteristic 0, which we discuss in Section 5.2. Finally, in Section 5.3, we finish the discussion of the second largest component, but only when $\text{char } k \neq 0$ (Corollary 5.11).

5.1 Restatement of the problem and the end of the proof

Lemma 5.1. *Let $[F] \in \mathbb{P}(V)$ be such that $\dim V(F)_{\text{sing}} \geq b$. Then there is an integral b -dimensional closed subscheme $C \hookrightarrow \mathbb{P}^n$ of degree at most $l(l-1)^{n+1}$ such that $C \subset V(F)_{\text{sing}}$.*

Proof. Let Z_1, \dots, Z_s be the irreducible components of $V(F)_{\text{sing}} = V(F, \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n})$. Then by Bézout's theorem ([1], p. 10),

$$\sum_{i=1}^s \deg(Z_i) \leq \deg(F) \prod_{\substack{0 \leq j \leq n \\ \partial F / \partial x_j \neq 0}} \deg\left(\frac{\partial F}{\partial x_j}\right) \leq l(l-1)^{n+1}.$$

But some component Z_i has dimension at least b , so, intersecting with hyperplanes if necessary, this component will contain an integral b -dimensional closed subscheme of degree at most $\deg(Z_i) \leq l(l-1)^{n+1}$. \square

Proof of Theorem 1.1 assuming the conjecture of Eisenbud and Harris. By Lemma 5.1, X is a finite union:

$$X = \bigcup_{d=1}^{l(l-1)^{n+1}} T_k^d. \quad (3)$$

In particular, X is a closed subset of $\mathbb{P}(V)$. The statement of Theorem 1.1 is now equivalent to the following one: for any $d \geq 2$, we have $\dim(T_k^d - T_k^1) < \dim X^1$. But $T_k^d - T_k^1 = (T^d - T^1)_k$, and if $k_0 \subset k$ is a subfield, then $\dim(T^d - T^1)_k = \dim(T^d - T^1)_{k_0}$. So it suffices to assume that $k = \overline{\mathbb{F}_p}$ or $k = \overline{\mathbb{Q}}$.

First, suppose that $k = \overline{\mathbb{F}_p}$. Let $\tau(l) = \lfloor \frac{l-1}{p} \rfloor$ and $m(l) = \lceil \frac{l+1}{2} \rceil$. Notice that $m(l) \geq \tau(l)+1$, so $m' = \tau(l)+1$ in Proposition 4.1. There exists $l_0 = l_0(n, b, p)$ (easily computable) such that for all $l \geq l_0$, we have $\binom{\tau(l)+b+1}{b+1} > a_{n,b}(l)$ and $A_b(\tau(l), \tau(l)+1) > a_{n,b}(l)$, by Remark 3.2 and the fact that $a_{n,b}(l)$ is a polynomial in l of degree b . We can assume in addition that l_0 satisfies Theorem 1.2. We claim that for any $l \geq l_0(n, b, p)$, the statement of Theorem 1.1 holds.

In fact, we prove by induction on $d \geq 2$ that for any irreducible component Z of T_k^d , either $Z = X^1$ or $\dim Z < \dim X^1$. For $2 \leq d \leq \frac{l+1}{2}$ this follows from Theorem 1.2. Let $d \geq \frac{l+1}{2}$. Assume that the statement holds for all $2 \leq d' \leq d-1$. Then it also holds for d , by Proposition 4.1, applied to the triple $(l, m(l), a_{n,b}(l)+1)$.

Now, let $k = \overline{\mathbb{Q}}$. Let p be any prime, and consider $l \geq l_0(n, b, p)$ as above. By the previous paragraph, for any $d \geq 2$, $\dim T_{\mathbb{F}_p}^d = \dim T_{\overline{\mathbb{F}_p}}^d \leq \dim X^1$. But, since $T^d \rightarrow \text{Spec } \mathbb{Z}$ is projective, by the upper semicontinuity theorem, we know

$$\dim T_{\overline{\mathbb{Q}}}^d = \dim T_{\mathbb{Q}}^d \leq \dim T_{\mathbb{F}_p}^d \leq \dim X^1.$$

Therefore, as long as $l \geq l_0(n, b, p)$ for some p (take $p = 2$ to obtain the best value of l_0 here), we know that X^1 (over $\overline{\mathbb{Q}}$) is an irreducible component of X (over $\overline{\mathbb{Q}}$) of maximal dimension.

We now address the question of uniqueness of X^1 as a largest component. In Section 5.2 we will show that it is possible to choose p such that $X^1 \not\subseteq T_{\mathbb{F}_p}^d$ for any $d \geq 2$. For such p , and for $d \geq 2$, the conclusion from two paragraphs ago implies

$$\dim T_{\mathbb{F}_p}^d < \dim X^1.$$

So

$$\dim T_{\overline{\mathbb{Q}}}^d \leq \dim T_{\mathbb{F}_p}^d < \dim X^1.$$

By (3), any irreducible component of X is either X^1 or is contained in T_k^d for some $d \geq 2$. This completes the proof. \square

Remark 5.2. We postpone for the next section the fact that over $\overline{\mathbb{F}_p}$, we have $X^1 \not\subseteq T_{\mathbb{F}_p}^d$, provided that $p \neq 2$ or $n - b$ is even. So for $l \geq l_0(n, b, 2)$, we know that X^1 is an irreducible component of X of largest dimension; for $l \geq l_0(n, b, 2)$ when $n - b$ is even, and for $l \geq l_0(n, b, 3)$ when $n - b$ is odd, we also know that X^1 is the unique largest-dimensional component of X .

Remark 5.3. We now give a (non-effective) proof of Theorem 1.1 without using the conjecture of Eisenbud and Harris. It is the same as before, except that we use Lemma 5.7 from Section 5.3 in [5] in place of Theorem 1.2, and use a different value of m in Proposition 4.1. Set $B = p^b(n - b + 1)$ in Lemma 5.7 in [5], and set $m = p^b(n - b + 1) + 1$ in Proposition 4.1. By the definition in Lemma 3.1 and by the definition of $\tau(l)$, we have that $A_b(\tau, m)$ grows as a polynomial in l of degree b and leading coefficient $\frac{m}{p^b b!} > \frac{n-b+1}{b!}$, so $A_b(\tau, m) > a_{n,b}(l)$ for sufficiently large l (recall the definition of $a_{n,b}(l)$ from the introduction). Thus, the hypothesis of Proposition 4.1 is satisfied again.

5.2 Uniqueness of the largest component (in characteristic 0)

We set the following notation for this section. Consider a b -dimensional closed subscheme $C = V(f, x_{b+2}, \dots, x_n)$ of \mathbb{P}^n , where $f \in k[x_0, \dots, x_{b+1}]_{d-\{0\}}$, and set $W = (f, x_{b+2}, \dots, x_n)_l^2$.

In [5], we proved the following

Lemma 5.4. *Assume $l \geq 2d + 1$. There is a dense open subset $U_1 \subset \mathbb{P}(W)$ such that for all $[F] \in U_1$, $V(F)_{\text{sing}} = C$ (set-theoretically).*

Next, we will use the lemma below only when C is linear, but we prove it here for a more general C for the purposes of the later discussion in Remark 5.12.

Lemma 5.5. *Suppose that $l \geq 2d$. If $\text{char } k \neq 2$, then there exists a dense open subset $U_2 \subset \mathbb{P}(W)$ such that for all $[F] \in U_2$, we have*

$$\dim\{P \in C \mid \dim T_P V(F)_{\text{sing}} \geq b + 1\} \leq b - 1.$$

If $\text{char } k = 2$ and C is a b -dimensional linear subspace, and $n - b$ is even, then the same conclusion holds.

Proof. Consider the incidence correspondence

$$Y_2 = \{([F], P) \in \mathbb{P}(W) \times C \mid \dim T_P V(F)_{\text{sing}} \geq b + 1\} \subset \mathbb{P}(W) \times C$$

(this is a closed subset). We will show that $Y_2 \neq \mathbb{P}(W) \times C$, i.e., $\dim Y_2 \leq \dim \mathbb{P}(W) + b - 1$. Once this is done, the map $Y_2 \rightarrow \mathbb{P}(W)$ will give a dense open $U_2 \subset \mathbb{P}(W)$ such that the fiber over any $[F] \in U_2$ has dimension at most $b - 1$.

Suppose that $\text{char } k \neq 2$. Fix a point $P = [p_0, \dots, p_{b+1}, 0, \dots, 0] \in C$ with at least 2 nonzero coordinates such that $V(f) \subset \mathbb{P}^{b+1} = V(x_{b+2}, \dots, x_n)$ is smooth at P . Without loss of generality, $\frac{\partial f}{\partial x_{b+1}}(P) \neq 0$ and $p_0 \neq 0$. We claim that there exists $[F] \in \mathbb{P}(W)$ with $\dim T_P V(F)_{\text{sing}} \leq b$.

For $[F] \in \mathbb{P}(W)$, we have $V(F)_{\text{sing}} = V(F, \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n})$, so we have to look at the Jacobian

$$J(P) = \begin{pmatrix} \frac{\partial F}{\partial x_0}(P) & \frac{\partial F}{\partial x_1}(P) & \dots & \frac{\partial F}{\partial x_n}(P) \\ \frac{\partial^2 F}{\partial x_0^2}(P) & \frac{\partial^2 F}{\partial x_0 \partial x_1}(P) & \dots & \frac{\partial^2 F}{\partial x_0 \partial x_n}(P) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 F}{\partial x_n \partial x_0}(P) & \frac{\partial^2 F}{\partial x_n \partial x_1}(P) & \dots & \frac{\partial^2 F}{\partial x_n^2}(P) \end{pmatrix}.$$

We know that $\dim T_P V(F)_{\text{sing}} = n - \text{rk } J(P)$, so $\dim T_P V(F)_{\text{sing}} \leq b$ if and only if $\text{rk } J(P) \geq n - b$. In other words, we have to give some $[F] \in \mathbb{P}(W)$ such that some $(n - b) \times (n - b)$ minor of the Jacobian is nonzero. Consider

$$F = x_0^{l-2d} f^2 + \sum_{i=b+2}^n x_0^{l-2} x_i^2.$$

We claim that the bottom right $(n - b) \times (n - b)$ minor of $J(P)$ is nonzero. Since $p_0 \neq 0$ and $\frac{\partial f}{\partial x_{b+1}}(P) \neq 0$,

$$\frac{\partial^2 F}{\partial x_{b+1}^2}(P) = 2x_0^{l-2d} \left(\left(\frac{\partial f}{\partial x_{b+1}} \right)^2 + f \frac{\partial^2 f}{\partial x_{b+1}^2} \right) (P) \neq 0,$$

so the minor

$$\left(\frac{\partial^2 F}{\partial x_i \partial x_j} (P) \right)_{b+1 \leq i, j \leq n} \quad (4)$$

is a diagonal matrix with nonzero diagonal entries.

Now suppose that $\text{char } k = 2$ but $n - b$ is even and $C = V(x_{b+1}, \dots, x_n)$. Let $P = [1, 0, \dots, 0]$. Consider $F = \sum_{i=1}^{\frac{n-b}{2}} x_{b+2i-1} x_{b+2i} x_0^{l-2}$. Then the minor (4) is nonzero again. \square

Remark 5.6. This lemma fails when $\text{char } k = 2$, C is linear, and $n - b$ is odd.

Corollary 5.7. *Suppose that $\text{char } k \neq 2$ or $\text{char } k = 2$ but $n - b$ is even. Then $X^1 \not\subseteq T_{\mathbb{F}_p}^d$ for any $d \geq 2$.*

Proof. Let $C = V(x_{b+1}, \dots, x_n)$. Let U_1 and U_2 be as given by Lemmas 5.4 and 5.5. Let $U = U_1 \cap U_2$. So U is a dense open subset of $P(W)$ such that for all $[F] \in U$, $V(F)_{\text{sing}} = L$ set-theoretically, and in addition, the closed embedding $L \hookrightarrow V(F)_{\text{sing}}$ is an isomorphism over the complement of a closed subset of smaller dimension. Thus the Hilbert polynomial of $V(F)_{\text{sing}}$ has degree b and leading term $1/b!$, so $V(F)_{\text{sing}}$ does not contain any closed subscheme of dimension b and degree $d \geq 2$. In other words, $[F] \in X^1 - T_{\mathbb{F}_p}^d$. \square

Similarly, we can apply Lemmas 5.4 and 5.5 to an integral $C = V(f, x_{b+2}, \dots, x_n)$ of degree 2 and obtain the following

Corollary 5.8. *Suppose that $\text{char } k \neq 2$. There exists $[F] \in \mathbb{P}(V)$ such that $V(F)_{\text{sing}}$ is a b -dimensional integral closed subscheme of degree 2 (as a set), and such that $V(F)_{\text{sing}}$ does not contain any b -dimensional closed subscheme of degree $d \geq 3$.*

5.3 The second largest component

We now determine the second largest component of X in the case $p = \text{char } k > 0$. In Section 6.1 in [5], we defined a certain irreducible closed subset X^2 of X with the property that if $[F] \in X$ contains an irreducible closed subscheme of dimension b and degree 2 in its singular locus, then $[F] \in X^2$, and we computed the dimension of X^2 . Also, in Section 6.2 in [5], we established the following

Lemma 5.9. *There exists l_0 (easily computable) such that for all pairs (d, l) with $3 \leq d \leq \frac{l+1}{2}$ and $l \geq l_0$ (if $b = n - 1$, assume $d \geq 4$), and any irreducible component Z of T_k^d , either $Z \subset T_k^1 \cup T_k^2$, or*

$$\dim Z < \binom{l+n}{n} - \dim X^2.$$

Remark 5.10. Lemma 5.9 does not treat the case $b = n - 1, d = 3$. We discuss this now. When $b = n - 1$, we can describe X explicitly. Indeed, if $V(G)$ is an integral $(n - 1)$ -dimensional closed subscheme of \mathbb{P}_k^n (here k has any characteristic) with $V(G) \subset V(F)_{\text{sing}}$, then necessarily $F = G^2 H$ for some H (since $V(G)$ is a complete intersection and the ideal (G^2) is saturated; see Corollary 2.3 in [6]). For $d = 1, \dots, \lfloor \frac{l}{2} \rfloor$, consider the map

$$\begin{aligned} \varphi_d: \mathbb{P}(k[x_0, \dots, x_n]_d) \times \mathbb{P}(k[x_0, \dots, x_n]_{l-2d}) &\longrightarrow \mathbb{P}(k[x_0, \dots, x_n]_l) \\ (G, H) &\longmapsto G^2 H. \end{aligned}$$

Certainly, $\text{im}(\varphi_d) \subset T_k^d \subset X$ and $X = \bigcup_{d=1}^{\lfloor \frac{l}{2} \rfloor} \text{im}(\varphi_d)$, so

$$X = X^1 \cup \text{im}(\varphi_2) \cup \text{im}(\varphi_3) \cup \left(\bigcup_{d=4}^{\lfloor \frac{l}{2} \rfloor} T^d \right).$$

Since any point in the image of φ_d has only finitely many preimages, it follows that

$$\dim \text{im}(\varphi_d) = \binom{d+n}{n} + \binom{l-2d+n}{n} - 2.$$

So $\dim \text{im}(\varphi_3) < \dim \text{im}(\varphi_2) = \dim X^2$ for $l \geq l_0$ (where l_0 is effectively computable) and hence when $b = n - 1$, it suffices to bound $\dim T_k^d$ only for $d \geq 4$, which was handled by Lemma 5.9.

Corollary 5.11. *Suppose that $\text{char } k = p > 0$. There exists (again, effectively computable) $l_0 = l_0(n, b, p)$ such that for all $l \geq l_0$, X^2 is the unique irreducible component of X of second largest dimension.*

Proof. Let $k = \overline{\mathbb{F}_p}$. With the above preparations, the proof is now analogous to that of Theorem 1.1. We use Lemma 5.9 (with Remark 5.10 if $b = n - 1$) and Proposition 4.1 to argue that if $Z \subset T_k^d$ is an irreducible component of T_k^d (where $d \geq 3$), then either $Z \subset T_k^1 \cup T_k^2$, or $\dim Z < \dim X^2$ (as long as $l \geq l_0$, for some effectively computable l_0).

We have

$$X = \bigcup_{d=1}^N T_k^d \quad \text{for } N = l(l-1)^{N+1}.$$

If Z is an irreducible component of X with $\dim Z \geq \dim X^2$, then $Z \subset T_k^d$ for some d . If $d \geq 3$, then by the previous paragraph, we have $Z \subset T^1 \cup T^2$. So in any case, $Z \subset T^1 \cup T^2 = X^1 \cup X^2$. Hence $Z = X^1$ or $Z = X^2$. \square

Remark 5.12. Let $p \neq 2$. If we could prove that $\dim T_{\overline{\mathbb{F}_p}}^d < \dim X^2$ for all $d \geq 3$, we would be able to deduce that for $d \geq 3$,

$$\dim T_{\mathbb{Q}}^d \leq \dim T_{\overline{\mathbb{F}_p}}^d < \dim X^2.$$

Suppose instead that $\dim T_k^d \geq \dim X^2$ for some $d \geq 3$ and $k = \overline{\mathbb{F}_p}$. Let Z be an irreducible component of T_k^d with $\dim Z \geq \dim X^2$. We have $Z \subset X^1 \cup X^2$ by the proof of Corollary 5.11. Moreover, $Z \not\subset X^2$ (since $X^2 \not\subset T_k^d$ by Corollary 5.8), so $Z \subset X^1$. So it would suffice to prove that $\dim(T_k^d \cap X^1) < \dim X^2$ for $d \geq 3$ (this inequality fails when $d = 2$). This is the technical problem that unfortunately does not allow us to remove the assumption $\text{char } k \neq 0$ from Corollary 5.11.

Acknowledgments

This work is part of my doctoral thesis at MIT, and I am gratefully indebted to my advisor Bjorn Poonen. He posed the problem whose solution we present here and guided me by suggesting various methods and approaches. In particular, the main idea involved in the current paper belongs to him. I admire Prof. Poonen's academic generosity.

References

- [1] William Fulton, *Introduction to intersection theory in algebraic geometry*, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1984.
- [2] Robin Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.
- [3] S. Lang, A. Weil, *Number of points of varieties in finite fields*, Amer. J. Math. **76** (1954), 819–827.
- [4] Bjorn Poonen, *Bertini theorems over finite fields*, Annals of Math. **160** (2004), no. 3, 1099–1127.
- [5] Kaloyan Slavov, *On the moduli space of hypersurfaces singular along a subscheme of large dimension but small degree*, arXiv:math/0527807v1.
- [6] Kaloyan Slavov, *The space of hypersurfaces singular along a specified curve*, arXiv:math/0527780v1.